UNKNOTTING NUMBER AND NUMBER OF REIDEMEISTER MOVES NEEDED FOR UNLINKING

CHUICHIRO HAYASHI AND MIWA HAYASHI

Abstract. Using unknotting number, we introduce a link diagram invariant of Hass and Nowik type, which changes at most by 2 under a Reidemeister move. As an application, we show that a certain infinite sequence of diagrams of the trivial two-component link need quadratic number of Reidemeister moves for being unknotted with respect to the number of crossings. Assuming a certain conjecture on unknotting numbers of a certain series of composites of torus knots, we show that the above diagrams need quadratic number of Reidemeister moves for being splitted.

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Keywords: link diagram, Reidemeister move, Hass-Nowik knot diagram invariant, unknotting number.

1. Introduction

In this paper, we regard that knot is a link with one component, and assume that links and link diagrams are oriented, and link diagrams are in the 2-sphere. A Reidemeister move is a local move of a link diagram as in Figure 1. An RI (resp. II) move creates or deletes a monogon face (resp. a bigon face). An RII move is called matched or unmatched according to the orientations of the edges of the bigon as shown in Figure 2. An RIII move is performed on a 3-gon face, deleting it and creating a new one. Any such move does not change the link type. As Alexander and Briggs [1] and Reidemeister [13] showed, for any pair of diagrams D_1, D_2 which represent the same link type, there is a finite sequence of Reidemeister moves which deforms D_1 to D_2.

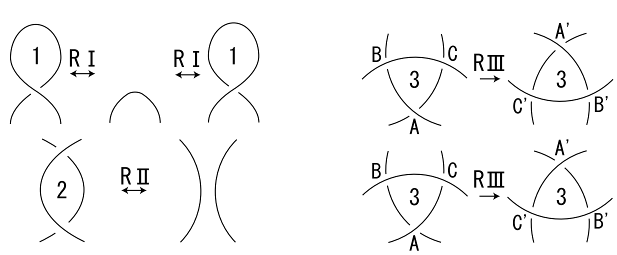

Figure 1.

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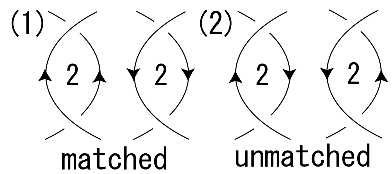

Figure 2.

Necessity of Reidemeister moves of type II and III is studied in [12], [10] and [3]. There are several studies of lower bounds for the number of Reidemeister moves connecting two knot diagrams of the same knot. See [4], [2], [7], [8], [5], [6]. In particular, Hass and Nowik introduced in [7] a certain knot diagram invariant I_{lk} by using the smoothing operation and the linking number. Let $\mathbb{G}_{\mathbb{Z}}$ be the free abelian group with basis $\{X_n, Y_n\}_{n\in\mathbb{Z}}$. The invariant I_{lk} assigns an element of $\mathbb{G}_{\mathbb{Z}}$ to a knot diagram. In [8], they showed that a certain homomorphism $g : \mathbb{G}_{\mathbb{Z}} \to \mathbb{Z}$ gives a numerical invariant $g(I_{lk})$ of a knot diagram which changes at most by one under a Reidemeister move. They gave an example of an infinite sequence of diagrams of the trivial knot such that the n-th one has $7n-1$ crossings, can be unknotted by $2n^2+3n$ Reidemeister moves, and needs at least $2n^2+3n-2$ Reidemeister moves for being unknotted.

The above papers studied Reidemeister moves on knot diagrams rather than link diagrams. In this paper, we introduce a link diagram invariant $iu(D)$ of Hass and Nowik type using unknotting number. The invariant $iu(D)$ changes at most by 2 under a Reidemeister move. As an application, we show that a certain infinite sequence of diagrams of the trivial two-component link need quadratic number of Reidemeister moves for being deformed into a diagram with no crossing. Assuming a certain conjecture on unknotting numbers of a certain series of composites of torus knots, we show that the above diagrams need quadratic number of Reidemeister moves for being splitted.

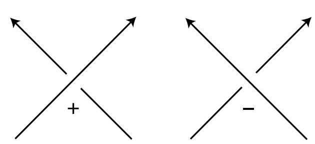

Figure 3.

We roughly sketch the definition of $iu(D)$. (Precise descriptions of the definitions of the unknotting number and $iu(D)$ are given in Section 2.) For a link L of m components, the *unknotting number* $u(L)$ of L is the X-Gordian distance between L and the trivial link of m components. We define the link diagram invariant $iu(D)$ as below. Let D be a diagram of an oriented link (possibly a knot) L_D. We assume that D is in the 2-sphere. For a crossing p of D, let D_p denote the link (possibly a knot) obtained from D by performing a smoothing operation at p with respect to the orientation of D. Note that D_p is a link rather than a

diagram. If L_D has m_D components, then D_p has m_D+1 components when p is a crossing between subarcs of the same component, and m_D-1 components when p is a crossing between subarcs of distinct component. Then we set $iu(D) = \sum_{p\in\mathcal{C}(D)} \mathrm{sign}(p)|\Delta u(D_p)|$, where $\mathcal{C}(D)$ is the set of all the crossings of D, and $\Delta u(D_p)$ is the difference between the unknotting numbers of D_p and L_D, i.e., $\Delta u(D_p) = u(D_p) - u(L_D)$. The sign of a crossing $\mathrm{sign}(p)$ is defined as in Figure 3 as usual. We set $iu(D) = 0$ for a diagram D with no crossing.

When D represents a knot, $iu(D) + w(D)$ with $w(D)$ being the writhe is the Hass-Nowik knot diagram invariant $g(I_\phi(D))$ introduced in [7] and [8] with g being the homomorphism with $g(X_k) = |k|+1$ and $g(Y_k) = -|k|-1$ as in [8], and ϕ being the difference of the unknotting numbers Δu.

Theorem 1.1. *The link diagram invariant $iu(D)$ does not change under an RI move and an unmatched RII move, and changes at most by one under a matched RII move, and at most by two under an RIII move.*

The above theorem is proved in Section 2.

Corollary 1.2. *Let D_1 and D_2 be link diagrams of the same oriented link. We need at least $|iu(D_1) - iu(D_2)|/2$ matched RII and RIII moves to deform D_1 to D_2 by a sequence of Reidemeister moves. In particular, when D_2 is a link diagram with no crossing, we need at leat $|iu(D_1)|/2$ matched RII and RIII moves.*

Note that, for estimation of the unknotting number, we can use the signature and the nullity (see Theorem 10.1 in [11] and Corollary 3.21 in [9]) or the sum of the absolute values of linking numbers over all pairs of components.

For a link diagram D, the sum of the signs of all the crossings is called the writhe and denoted by $w(D)$. It does not change under an RII or RIII move but increases (resp. decreases) by 1 under an RI move creating a positive (resp. negative) crossing. Set $iu_{\epsilon,\delta}(D) = iu(D) + \epsilon(\frac{1}{2}c(D) + \delta\frac{3}{2}w(D))$ for a link diagram D, where $\epsilon = \pm 1$, $\delta = \pm 1$ and $c(D)$ denotes the number of crossings of D. Then we have the next corollary.

Corollary 1.3. *The link diagram invariant $iu_{\epsilon,+1}(D)$ (resp. $iu_{\epsilon,-1}(D)$) increases by 2ϵ under an RI move creating a positive (resp. negative) crossing, decreases by ϵ under an RI move creating a negative (resp. positive) crossing, increases by ϵ under an unmatched RII move, changes at most by 2 under a matched RII move, and changes at most by 2 under an RIII move.*

Let D_1 and D_2 be link diagrams of the same oriented link. We need at least $|iu_{\epsilon,\delta}(D_1) - iu_{\epsilon,\delta}(D_2)|/2$ Reidemeister moves to deform D_1 to D_2. In particular, when D_2 is a link diagram with no crossing, we need at least $|iu_{\epsilon,\delta}(D_1)|/2$ Reidemeister moves.

Remark 1.4. We can set $iu'(D) = \sum_{p \in \mathcal{S}(D)} \mathrm{sign}(p)|\Delta u(D_p)| + \sum_{p \in \mathcal{M}(D)} \mathrm{sign}(p) \cdot u(D_p)$, where $\mathcal{S}(D)$ denotes the all crossings of D between subarcs of the same component, and $\mathcal{M}(D)$ denotes the all crossings of D between subarcs of distinct components. Then $iu'(D)$ has the same properties as those of $iu(D)$ described in Theorem 1.1, Corollary 1.2 and Corollary 1.3.

Suppose that an m-component link L is split, and there is a splitting 2-sphere which separates components $J_1, J_2, \cdots, J_k$ of L form the other components $K_1, K_2, \cdots, K_\ell$ of L. Set $J = \{J_1, \cdots, J_k\}$, and $K = \{K_1, \cdots, K_\ell\}$. Let D be a diagram of L in the 2-sphere. We denote by $\mathcal{C}(J, K, D)$ the set of all crossings of D between a subarc of J_i and a subarc of K_j for some $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Then we set $iu'(J, K, D) = \sum_{p \in \mathcal{C}(J,K,D)} \mathrm{sign}(p) \cdot u(D_p)$. If $\mathcal{C}(J, K, D) = \emptyset$, then we set $iu'(J, K, D) = 0$.

A similar argument as the proof of Theorem 1.1 shows the next theorem. We omit the proof.

Theorem 1.5. *Let L, J, K, D be as above. The link diagram invariant $iu'(J, K, D)$ does not change under an RI move and an unmatched RII move, and changes at most by one under a matched RII move, and at most by two under an RIII move. We need at least $|iu'(J, K, D)|/2$ matched RII and RIII moves to deform D to a split link diagram E with $\mathcal{C}(J, K, E) = \emptyset$ by a sequence of Reidemeister moves.*

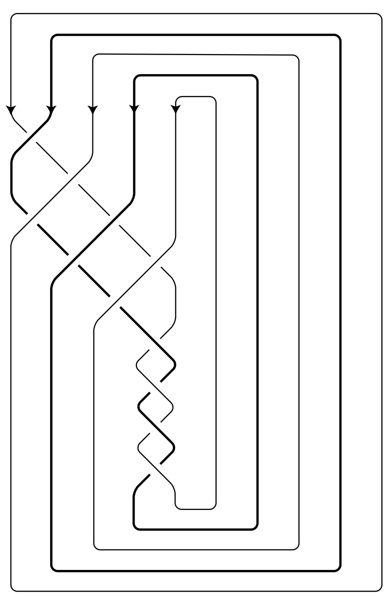

FIGURE 4.

Let us describe the link diagram D_n with n being a natural number. See Figure 4, where D_n with $n = 4$ is depicted. For any $i \in \{1, 2, \cdots, n-1\}$, let σ_i be the generator of the n-braid group B_n, which denotes the braid where the i-th strand crosses over the $(i+1)$ st strand

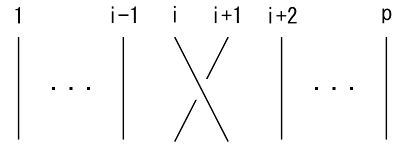

FIGURE 5.

(Figure 5). Then, D_n is the closure of the $(n+1)$-braid $\sigma_1^{-1}(\sigma_2^{-1}\sigma_1^{-1})(\sigma_3^{-1}\sigma_2^{-1})\cdots(\sigma_n^{-1}\sigma_{n-1}^{-1})\sigma_n^n$. We orient D_n so that it is descending on the braid. Thus D_n has $2n-1$ positive crossings and n negative crossings.

Theorem 1.6. *For any natural number n, the diagram D_n of the trivial two-component link can be deformed to a diagram with no crososing by a sequence of $(n^2+3n-2)/2$ Reidemeister moves which consists of $n-1$ RI moves deleting a positive crossing, n matched RII moves deleting a bigon face, and $(n-1)n/2$ RIII moves. Moreover, any sequence of Reidemeister moves bringing D_n to a diagram with no crossing must contain $\frac{1}{2}[3n-2+2\sum_{k=1}^{n-1} u(T(2,k)\sharp T(2,-k))+u(T(2,n)\sharp T(2,-n))]$ or larger number of Reidemeister moves, where $T(2,k)$ is the $(2,k)$-torus link, $T(2,-k)$ is the mirror image of $T(2,k)$, and $\sharp$ denotes the connected sum.*

We estimate the sum $\Sigma = \sum_{k=1}^{n-1} u(T(2,k)\sharp T(2,-k)) + u(T(2,n)\sharp T(2,-n))$. For an even number k, the link $T(2,k)\sharp T(2,-k)$ has 3 components. By using the sum of the absolute values of the linking numbers, we can see easily that $u(T(2,k)\sharp T(2,-k)) = k$. For an odd number k larger than 1, $T(2,k)\sharp T(2,-k)$ is a composite knot. A composite knot has 2 or greater unknotting number, which was shown in [14] by M. Scharlemann. Hence we have $\Sigma \geq (n^2+4n-8)/2$ when n is even, and $\Sigma \geq (n^2+4n-9)/2$ when n is odd. If the conjecture on the unknotting number below holds, then $\Sigma = n^2 - n$.

Conjecture 1.7. $u(T(2,k)\sharp T(2,-k)) = k-1$ for any odd integer k.

Applying Theorem 1.5 to the diagram D_n, we have the next theorem. We omit the proof.

Theorem 1.8. *Any sequence of Reidemeister moves bringing D_n to a disconnected diagram must contain $\sum_{k=1}^{\frac{n}{2}-1} u(T(2,2k+1)\sharp T(2,-(2k+1)))$ or larger number of Reidemeister moves when n is even, and $\frac{1}{2}[2\sum_{k=1}^{\frac{n-1}{2}-1} u(T(2,2k+1)\sharp T(2,-(2k+1)))+u(T(2,n)\sharp T(2,-n))]$ or larger number of Reidemeister moves when n is odd.*

Since the unknotting number of a composite knot is greater than or equal to 2, the above number is larger than or equal to $n-2$. If Conjecture 1.7 is true, then the above number is equal to $(n^2-2n)/4$ when n is even, and to $(n^2-2n+1)/4$ when n is odd.

The precise definition of link diagram invariant $iu(D)$ is given in Section 2, where changes of the value of the invariant under Reidemeister moves are studied. The sequence of Reidemeister moves in Theorem 1.6 is described in Section 3. In Section 4, we calculate $iu(D_n)$, to prove Theorem 1.6.

2. LINK DIAGRAM INVARIANT

A link is called the *trivial n-component link* if it has n components and bounds a disjoint union of n disks. The trivial n-component link admits a *trivial diagram* with no crossing.

Let L be a link with n components, and D a diagram of L. We call a sequence of Reidemeister moves and crossing changes on D an *X-unknotting sequence* in this paragraph, if it deforms D into a (possibly non-trivial) diagram of the trivial n-component link. The *length* of an X-unknotting sequence is the number of crossing changes in it. The minimum length among all the X-uknotting sequences on D is called the *uknotting number* of L. We denote it by $u(L)$. Clearly, $u(L)$ depends only on L and not on D.

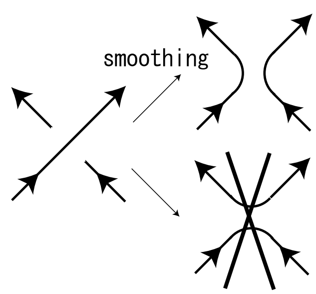

FIGURE 6.

Let D be an oriented link diagram in the 2-sphere, L_D the link represented by D, p a crossing of D, and D_p the link (rather than a diagram) obtained from D by performing the smoothing operation on D at p as below. We first cut the link at the two preimage points of p. Then we obtain the four endpionts. We paste the four short subarcs of the link near the endpoints in the way other than the original one so that their orientations are connected consistently. See Figure 6.

We set $iu(D)$ to be the sum of the absolute value of the difference of the unknotting numbers $\Delta u(D_p) = u(D_p) - u(L_D)$ with the sign of p over all the crossings of D, i.e.,

$$iu(D) = \sum_{p \in \mathcal{C}(D)} \mathrm{sign}(p) \cdot |\Delta u(D_p)|$$

where $\mathcal{C}(D)$ is the set of all the crossings of D. For a diagram D with no crossing, we set $iu(D) = 0$. Note that, when D represents a knot, $iu(D) + w(D)$ with w being the writhe is equal to Hass-Nowik invariant $g(I_\phi(D))$ with ϕ being Δu and g being the homomorphism with $g(X_k) = |k| + 1$ and $g(Y_k) = -|k| - 1$ (see [7] and [8]).

Theorem 2.1. *$iu(D)$ does not change under an RI move and an unmatched RII move, changes at most by one under a matched RII move, and at most by two under an RIII move.*

Proof. The proof is very similar to the arguments in Section 2 in [7]. Let D, E be link diagrams such that E is obtained from D by a Reidemeister move. Let L_D, L_E be the links represented by D, E respectively. Note that L_D and L_E are the same link.

First, we suppose that E is obtained from D by an RI move creating a crossing a. Then the link E_a differs from L_E by an isolated single trivial component, and hence $u(E_a) = u(L_E)$. Then the contribution of a to iu is $\pm(u(E_a) - u(L_E)) = 0$. The contribution of any other crossing x to iu is unchanged since the RI move shows that L_D and L_E are the same link and so D_x and E_x are. Thus an RI move does not change iu, i.e., $iu(D) = iu(E)$.

When E is obtained from D by an RII move creating a bigon face, let x and y be the positive and negative crossings at the corners of the bigon. If the RII move is unmatched, then E_x and E_y are the same link. Hence $u(E_x) = u(E_y)$ and $|iu(E) - iu(D)| = ||u(E_x) - u(L_E)| - |u(E_y) - u(L_E)|| = 0$. If the RII move is matched, then E_x and E_y differ by a crossing change, and hence their unknotting numbers differ by at most one, i.e., $|u(E_x) - u(E_y)| \leq 1$. Hence $|iu(E) - iu(D)| = ||u(E_x) - u(L_E)| - |u(E_y) - u(L_E)|| \leq |(u(E_x) - u(L_E)) - (u(E_y) - u(L_E))| = |u(E_x) - u(E_y)| \leq 1$.

We consider the case where E is obtained from D by an RIII move. For the crossing x between the top and the middle strands of the trigonal face where the RIII move is applied, D_x and E_x are the same link. Hence the contribution of x to iu is unchanged. The same is true for the crossing y between the bottom and the middle strands. Let z be the crossing between the top and the bottom strands. Then D_z and E_z differ by two crossing changes and Redemeister moves, and hence $|u(E_z) - u(D_z)| \leq 2$. Thus $|iu(E) - iu(D)| = ||u(E_z) - u(L_E)| - |u(D_z) - u(L_D)|| \leq |(u(E_z) - u(L_E)) - (u(D_z) - u(L_D))| = |u(E_z) - u(D_z)| \leq 2$. □

3. Unknotting sequence of Reidemeister moves on D_n

In this section, we deform the link diagram D_n to a diagram with no crossing by a sequence of Reidemeister moves.

Lemma 3.1. *The closure of the braid $b = \sigma_1^{-k}(\sigma_2^{-1}\sigma_1^{-1})(\sigma_3^{-1}\sigma_2^{-1})\cdots(\sigma_m^{-1}\sigma_{m-1}^{-1})\sigma_m^{\ell}$ can be deformed into that of $b' = \sigma_1^{-k-1}(\sigma_2^{-1}\sigma_1^{-1})(\sigma_3^{-1}\sigma_2^{-1})\cdots(\sigma_{m-1}^{-1}\sigma_{m-2}^{-1})\sigma_{m-1}^{\ell}$ by a sequence of k RIII moves and a single RI move.*

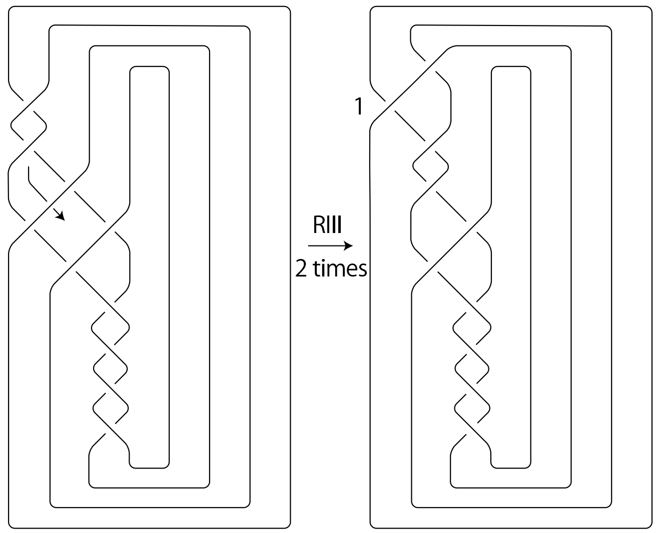

FIGURE 7.

Proof. Applying the braid relation $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ repeatedly k times, we obtain the closed braid of $\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-k}(\sigma_3^{-1}\sigma_2^{-1})\cdots(\sigma_m^{-1}\sigma_{m-1}^{-1})\sigma_m^{\ell}$. This is accomplished by a sequence of k RIII moves. See Figure 7. Thus we can apply RI move (Markov's destabilization) on the outermost region which is a monogon face, to obtain b'. Note that this remove the only σ_1^{-1}, and reduces the suffix numbers of σ_i $(i \geq 2)$ by one. □

We can apply the deformation in Lemma 3.1 repeatedly $n-1$ times to deform D_n into the closure of the 2-braid $\sigma_1^{-n}\sigma_1^n$. Then a sequence of n matching RII moves deletes all the crossings. If n is a natural number, then this deformation consists of $n-1$ RI moves deleting a positive crossing, n matched RII moves deleting a bigon and $1+2+\cdots+(n-1) = (n-1)n/2$ RIII moves.

Thus the former half of Theorem 1.6 holds.

4. Calculation of $iu(D_n)$ and proof of Theorem 1.6

Lemma 4.1. $iu(D_n) = 2\sum_{k=1}^{n-1} u(T(2,k) \sharp T(2,-k)) + u(T(2,n) \sharp T(2,-n))$,

and $iu_{+1,+1}(D_n) = 3n-2+2\sum_{k=1}^{n-1} u(T(2,k) \sharp T(2,-k)) + u(T(2,n) \sharp T(2,-n))$.

Proof. If we perform a smoothing at a crossing of D_n corresponding to σ_k^{-1} for $1 \leq k \leq n$, then we obtain the link $T(2,k) \sharp T(2,-k)$. See (1) in Figure 8. A smoothing operation at a crossing of D_n corresponding to σ_n yields the trivial knot. See (2) in Figure 8. Since D_n represents the trivial 2-component link, $\Delta u(D_x) = u(D_x)$ for any crossing x of D_n. Hence we obtain the above formula of $iu(D_n)$. Note that there are two crossing points of

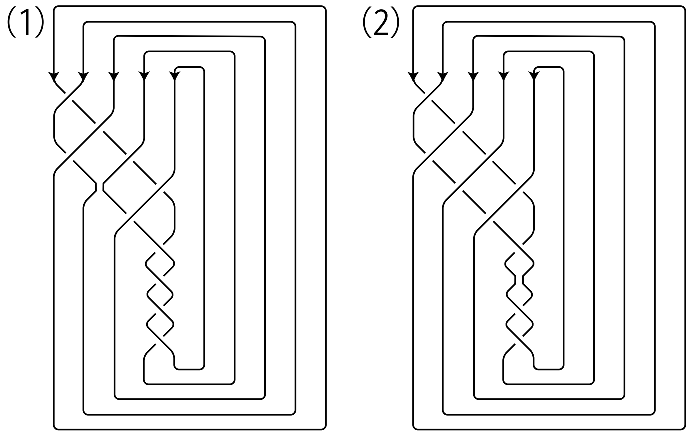

FIGURE 8.

D_n corresponding to σ_k^{-1} with $1 \leq k \leq n-1$, while there is only one crossing point of D_n corresponding to σ_n^{-1}.

Since D_n has $2n-1$ positive crossings and n negative crossings, $c(D_n)/2 + 3w(D_n)/2 = ((2n-1)+n)/2 + 3((2n-1)-n)/2 = 3n-2$. Thus we obtain the above formula of $iu_{+1,+1}(D_n)$. □

Proof of Theorem 1.6. The former half of Theorem 1.6 is already shown in Section 3. The above formula of $iu_{+1,+1}(D_n)$ and Corollary 1.3 together show the latter half of Theorem 1.6. □

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Chuichiro Hayashi: Department of Mathematical and Physical Sciences, Faculty of Science, Japan Women's University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo, 112-8681, Japan.
hayashic@fc.jwu.ac.jp

Miwa Hayashi: Department of Mathematical and Physical Sciences, Faculty of Science, Japan Women's University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo, 112-8681, Japan.
miwakura@fc.jwu.ac.jp